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New realizations of Lie algebra kappa-deformed Euclidean space

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Abstract. We study Lie algebra κ-deformed Euclidean space with undeformed rotation algebra $SO_a(n)$ and commuting vectorlike derivatives. Infinitely many realizations in terms of commuting coordinates are constructed and a corresponding star product is found for each of them. The κ-deformed noncommutative space of the Lie algebra type with undeformed Poincaré algebra and with the corresponding deformed coalgebra is constructed in a unified way.

1 Introduction

In the last decade, there has been a great interest in the formulation and consistency of physical theories defined on noncommutative (NC) spaces, and in finding their consequences [1–5].

However, there is no clear guiding physical principle for how to build a fundamental theory on NC spaces and which NC spaces are physically acceptable and preferable. Also, it is not known how matter and gravity influence the properties of NC spaces at small distances, and vice versa.

Nevertheless, it is important to classify NC spaces and their properties, and, particularly, to develop a unifying approach to and a generalized theory for such NC spaces that are convenient for physical applications. The notion of generalized symmetries and their role in the analysis of NC spaces is also crucial. In order to make a step in this direction, we analyze a NC space of the Lie algebra type, particularly the so-called κ-deformed space introduced in [6–8].

For simplicity, we restrict ourselves to κ-deformed Euclidean space. The analysis can be easily extended to κ-deformed Minkowski space. The dimensional parameter $a = \frac{1}{\kappa}$ is a very small length scale, and when it goes to zero, the undeformed space appears as a smooth limit. The generators of generalized rotations satisfy the undeformed $SO_a(n)$ algebra, i.e. the undeformed Lorentz algebra in κ-deformed Minkowski space. Dirac derivatives are assumed to mutually commute and transform as a vector representation under $SO_a(n)$ algebra. This κ-deformed space was studied by different groups, from both the mathematical and physical point of view [9–18]. Specially, realizations in terms of commutative coordinates were obtained and discussed in the cases of symmetric ordering and normal (left and right) ordering of NC coordinates [12, 18].

We analyze κ-deformed Euclidean space using the methods developed for deformed single and multimode oscillators in the Fock space representations [19–26]. Particularly, we use the methods for constructing deformed creation and annihilation operators in terms of ordinary bosonic multimode oscillators, i.e. a kind of bosonisation [19, 20, 25]. Also, we use the construction of transition number operators and, generally, of generators proposed in [20, 21, 24].

The simple connection between creation and annihilation operators with NC coordinates and Dirac derivatives is established by the Bargmann-type representation. We find infinitely many new realizations in terms of commutative coordinates. All these realizations are on an equal footing, and a star product is associated to each of them. The general feature of NC spaces is that there are generally infinitely many realizations in terms of commutative coordinates and the physical results should not depend on the realization used.

The plan of the paper is as follows. In Sect. 2 we present κ-deformed Euclidean space and its realizations in Euclidean space. In Sect. 3 the undeformed rotation algebra $SO_a(n)$ compatible with kappa deformation and its general realizations are considered. In Sect. 4 the action of the generators $SO_a(n)$ on NC coordinates leads to infinitely many new realizations of NC space in terms of commutative coordinates and their derivatives. In Sect. 5 the corresponding realizations of Dirac derivatives are constructed. The Lie algebra type κ-deformed NC space with undeformed rotation (Poincaré) algebra and the corresponding deformed coalgebra is proposed in a unique way. In Sect. 6 the invariant Klein–Gordon operator and its realizations are given for κ-deformed Euclidean space, with a short summary of all realizations included. In Sect. 7 the hermiticity properties are discussed. In Sect. 8 the corresponding realizations for star products are presented. Finally, in Sect. 9 a short conclusion is given.

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2 Kappa-deformed Euclidean space and its realizations

Let us consider a Lie algebra type noncommutative (NC) space with coordinates $\hat{x}_1, \hat{x}_2, ..., \hat{x}_n$, as follows:

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}C_{\mu\nu\lambda}\hat{x}_\lambda = \mathrm{i}(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)\,, \tag{1}$$

where $a_1, a_2, ..., a_n$ are constant real parameters describing a deformation of Euclidean space. The structure constants are

$$C_{\mu\nu\lambda} = a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}\,. \tag{2}$$

We choose $a_1 = a_2 = ... = a_{n-1} = 0, a_n = a$ and use Latin indices for the subspace $(1, 2, ..., n-1)$ and Greek indices for the whole space $(1, 2, ..., n)$ [16–18]. Then the algebra of the NC coordinates becomes

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{x}_n, \hat{x}_i] = \mathrm{i}a\hat{x}_i, \qquad i, j = 1, 2, ..., n-1\,. \tag{3}$$

Using the methods developed in [19, 20, 24, 25] and the Bargmann representation we point out that there exists a realization of the NC coordinates $\hat{x}_\mu$ in terms of ordinary commutative coordinates $x_1, x_2, ..., x_n$ and their derivatives $\partial_1, \partial_2, ..., \partial_n$, where $\partial_\mu = \frac{\partial}{\partial x_\mu}$. The general Ansatz for the NC coordinates $\hat{x}_\mu$ satisfying the algebra (3) is

$$\begin{aligned} \hat{x}_i &= x_i \varphi(A)\,, \\ \hat{x}_n &= x_n \psi(A) + \mathrm{i}a x_k \partial_k \gamma(A)\,, \end{aligned} \tag{4}$$

where $A = \mathrm{i}a\partial_n$. In the above relations the deformed creation operators are represented by $\hat{x}_\mu$, the bosonic creation operators by x_μ, the bosonic annihilation operator by ∂_μ and the vacuum state by 1. Inserting this Ansatz into (3) we obtain

$$\frac{\varphi'}{\varphi}\psi = \gamma - 1\,, \tag{5}$$

where $\varphi' = \frac{\mathrm{d}\varphi}{\mathrm{d}A}$. There are infinitely many representations parametrized by two of the functions φ, ψ, γ, with the boundary conditions

$$\varphi(0) = 1, \quad \psi(0) = 1\,, \tag{6}$$

and with $\gamma(0) = \varphi'(0) + 1$ finite. At this point we could make rather ad hoc arbitrary assumptions on the derivatives $\hat{\partial}_\mu$ which cannot be motivated physically nor mathematically.

3 $SO_a(n)$ algebra

Instead, we demand that there should exist generators $M_{\mu\nu}$ satisfying the ordinary undeformed $SO_a(n)$ algebra:

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda}M_{\mu\rho} - \delta_{\mu\lambda}M_{\nu\rho} - \delta_{\nu\rho}M_{\mu\lambda} + \delta_{\mu\rho}M_{\nu\lambda}\,. \tag{7}$$

There are infinitely many representations of $M_{\mu\nu}$ in terms of commutative coordinates x_λ, their derivatives ∂_λ, and deformation parameters $a_1, a_2, ..., a_n$. The generators $M_{\mu\nu}$ are linear in x and form an infinite series in ∂.

Let us assume that $a_i = 0$, $a_n = a$, i.e. the same deformation parameters as in the $\hat{x}$ NC coordinate algebra, (3). Then a simple Ansatz is

$$\begin{aligned} M_{ij} &= x_i \partial_j - x_j \partial_i\,, \\ M_{in} &= x_i \partial_n F_1 - x_n \partial_i F_2 + \mathrm{i}a x_i \Delta F_3 + \mathrm{i}a x_k \partial_k \partial_i F_4\,, \\ M_{ni} &= -M_{in}\,, \end{aligned} \tag{8}$$

where $\Delta = \partial_k \partial_k$ and the summation over repeated indices is understood. The functions F_1, F_2, F_3, F_4 depend on $A = \mathrm{i}a\partial_n$. In principle, the F functions could depend on $B = (\mathrm{i}a)^2\Delta$. For simplicity, we assume that F dependen on A only. Inserting Ansatz (8), into the algebra (7), from

$$\begin{aligned} [M_{ij}, M_{jn}] &= M_{in}\,, \\ [M_{in}, M_{jn}] &= -M_{ij}\,, \end{aligned} \tag{9}$$

we obtain the following two equations:

$$\begin{aligned} &F_1 F_2 + A\, F_1' F_2 + A F_1 F_4 - 2A F_1 F_3 = 1\,, \\ &2F_3^2 - F_3' F_2 + F_3 F_4 = 0\,, \end{aligned} \tag{10}$$

where $F' = \frac{\mathrm{d}F}{\mathrm{d}A}$. Note that two of the functions F_1, F_2, F_3, F_4 are arbitrary. The boundary conditions are

$$F_1(0) = F_2(0) = 1\,, \tag{11}$$

and $F_3(0)$, $F_4(0)$ are required to be finite.

Now we can calculate the commutators $[M_{\mu\nu}, \hat{x}_\lambda]$, substituting $\hat{x}_\lambda$, (4), and $M_{\mu\nu}$, (8). The result is expressed in terms of φ, ψ, γ and F functions, restricted by (5) and (10), and is linear in the commutative coordinates x_μ. In general, this result cannot be expressed in terms of $\hat{x}$ and M only, without the derivatives ∂.

4 Action of $SO_a(n)$ generators on NC coordinates

At this point we demand that the generators $M_{\mu\nu}$, $\hat{x}_\lambda$ close linearly under commutation, i.e. we obtain the extended Lie algebra with extended structure constants satisfying Jacobi relations. In order to construct the extended Lie algebra of generators the $\hat{x}_\lambda$, and $M_{\mu\nu}$, for general a_μ, we proceed as follows. The most general covariant form of commutators $[M_{\mu\nu}, \hat{x}_\lambda]$, of the generators of rotations $M_{\mu\nu}$ with NC coordinates $\hat{x}_\lambda$, is antisymmetric in the indices μ and ν, linear in the generators $\hat{x}$, M, and with smooth limit $[M_{\mu\nu}, x_\lambda] = x_\mu \delta_{\nu\lambda} - x_\nu \delta_{\mu\lambda}$, when $a_\mu \to 0$. It is given by

$$\begin{aligned} [M_{\mu\nu}, \hat{x}_\lambda] = {} & \hat{x}_\mu \delta_{\nu\lambda} - \hat{x}_\nu \delta_{\mu\lambda} + \mathrm{i}s a_\lambda M_{\mu\nu} \\ & - \mathrm{i}t(a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}) \\ & + \mathrm{i}u a_\alpha (M_{\alpha\mu}\delta_{\nu\lambda} - M_{\alpha\nu}\delta_{\mu\lambda})\,, \end{aligned}$$

where $s, t, u \in \mathbf{R}$.

The necessary and sufficient conditions for the consistency of the extended Lie algebra with generators $\hat{x}_\lambda$ and $M_{\mu\nu}$ are

$$[M_{\alpha\beta},[\hat{x}_\mu,\hat{x}_\nu]]+[\hat{x}_\mu,[\hat{x}_\nu,M_{\alpha\beta}]]+[\hat{x}_\nu,[M_{\alpha\beta},\hat{x}_\mu]]=0\,,$$
$$[M_{\alpha\beta},[M_{\gamma\delta},\hat{x}_\mu]]+[M_{\gamma\delta},[\hat{x}_\mu,M_{\alpha\beta}]]+[\hat{x}_\mu,[M_{\alpha\beta},M_{\gamma\delta}]]=0\,.$$

All extended Jacobi identities are satisfied for the unique solution $s=u=0, t=1$:

$$[M_{\mu\nu},\hat{x}_\lambda]=\hat{x}_\mu\delta_{\nu\lambda}-\hat{x}_\nu\delta_{\mu\lambda}-ia_\mu M_{\nu\lambda}+ia_\nu M_{\mu\lambda}\,. \quad (12)$$

Inserting $a_i=0, a_n=a$ we obtain two important relations:

$$[M_{in},\hat{x}_n]=\hat{x}_i+\mathrm{i}aM_{in}\,, \quad (13)$$

$$[M_{in},\hat{x}_j]=-\delta_{ij}\hat{x}_n+\mathrm{i}aM_{ij}\,. \quad (14)$$

An important ingredient of the symmetry structure of κ-deformed space are the Leibniz rules of the generators of the rotations. They can be derived immediately from (13) and (14) [16–18]

$$M_{ij}(f\cdot g)=(M_{ij}f)\cdot g+f\cdot(M_{ij}g)\,,$$
$$M_{in}(f\cdot g)=(M_{in}f)\cdot g+(\mathrm{e}^A f)\cdot(M_{in}g)-\mathrm{i}a\left(\partial_j\frac{1}{\varphi(A)}f\right)\cdot(M_{ij}g)\,,$$

where f, g are functions of the NC coordinates $\hat{x}_\mu$, and

$$[\partial_i,\hat{x}_j]=\delta_{ij}\varphi(A)\,,\qquad [\partial_i,\hat{x}_n]=\mathrm{i}a\partial_i\gamma(A)\,,$$
$$[\partial_n,\hat{x}_i]=0\,,\qquad [\partial_n,\hat{x}_n]=1\,,$$

for $\psi=1$.

In a more technical language, the above equations are the coproducts

$$\triangle M_{ij}=M_{ij}\otimes 1+1\otimes M_{ij}\,,$$
$$\triangle M_{in}=M_{in}\otimes 1+\mathrm{e}^A\otimes M_{in}-\mathrm{i}aD_j\mathrm{e}^A\otimes M_{ij}\,,$$

where e^A and the Dirac derivatives D_μ are defined in Sect. 5, (29), for the case $\psi=1$. The final result for the coproduct depends only on aD_μ, and $a^2D_\mu D_\mu$. In the limit $a\to 0$ it gives ordinary undeformed coproduct for $M_{\mu\nu}$. The coproduct Δ, which we determined for $M_{\mu\nu}$, multiplicatively extends to the whole algebra $SO_a(n)$, which becomes a Hopf algebra in this way.

From (13) we obtain four equations:

$$F_1\psi+AF_1'\psi-AF_1(\gamma+1)-\varphi=0\,,$$
$$F_2'\psi-F_2\psi'+F_2(\gamma-1)=0\,,$$
$$F_3'\psi+F_3(\gamma-1)=0\,,$$
$$F_4'\psi+F_2\gamma'+F_4(\gamma-1)=0\,, \quad (15)$$

and from (14) we obtain two equations:

$$F_2\varphi'+F_4\varphi+1=0\,,$$
$$F_3=\frac{1}{2\varphi}\,. \quad (16)$$

Now we have six additional equations, i.e. eight equations (10), (15), (16) for four functions F_1, F_2, F_3 and F_4. Hence, there are four additional equations which have to be satisfied. From these consistency relations we obtain two infinite families of solutions satisfying simultaneously (10), (15) and (16).

I realization: $\psi=1$

We have

$$F_1=\varphi\frac{\mathrm{e}^{2A}-1}{2A}\,,\quad F_2=\frac{1}{\varphi}\,,\quad F_3=\frac{1}{2\varphi}\,,\quad F_4=-\frac{\gamma}{\varphi}\,, \quad (17)$$

where

$$\gamma=\frac{\varphi'}{\varphi}+1\,. \quad (18)$$

II realization: $\psi=1+2A$

We have

$$F_1=\varphi\,,\quad F_2=\frac{\psi}{\varphi}\,,\quad F_3=\frac{1}{2\varphi}\,,\quad F_4=-\frac{\gamma}{\varphi}\,, \quad (19)$$

where

$$\varphi=\frac{C-1}{C-\sqrt{\psi}}\,,\quad \gamma=\psi\frac{\varphi'}{\varphi}+1\,,\quad C\in\mathbf{R}\,,\quad C\neq 1\,. \quad (20)$$

The first realization $\psi=1$ can be parametrized by an arbitrary function $\varphi(A)$, $\varphi(0)=1$. The second realization $\psi=1+2A$ is parametrized with $C\in\mathbf{R}$, $C\neq 1$.

5 Dirac derivatives

Imposing the undeformed $SO_a(n)$ algebra it is natural to define the Dirac derivatives D_μ as

$$[M_{\mu\nu},D_\lambda]=\delta_{\nu\lambda}D_\mu-\delta_{\mu\lambda}D_\nu\,,$$
$$[D_\mu,D_\nu]=0\,. \quad (21)$$

The most general Ansatz corresponding to $a_i=0, a_n=a$ is

$$D_i=\partial_iG_1(A)\,,$$
$$D_n=\partial_nG_2(A)+\mathrm{i}a\Delta G_3(A)\,. \quad (22)$$

Inserting them into

$$[M_{in},D_n]=D_i\,,$$
$$[M_{in},D_i]=-D_n\,, \quad (23)$$

we find four equations:

$$AF_2G_2'+F_2G_2-2AF_1G_3-G_1=0\,,$$
$$F_2G_3'-2(F_3+F_4)G_3=0\,,$$
$$F_1G_1-G_2=0\,,$$
$$2F_2G_1'-2(F_3+F_4)G_1+2G_3=0\,. \quad (24)$$

The boundary conditions are

$$G_1(0)=1\,,\quad G_2(0)=1\,, \tag{25}$$

and $G_3(0)$ is required to be finite. Using our realizations for the F functions, (17)–(20), we find the following.

I realization: $\psi=1$

We have

$$G_1=\frac{\mathrm{e}^{-A}}{\varphi}\,,\quad G_2=\frac{\sinh A}{A}\,,\quad G_3=\frac{\mathrm{e}^{-A}}{2\varphi^2}\,. \tag{26}$$

II realization: $\psi=1+2A$

We have

$$G_1=\frac{C-\sqrt{\psi}}{(C-1)\sqrt{\psi}}\,,\quad G_2=\frac{1}{\sqrt{\psi}}\,,\quad G_3=\frac{\left(C-\sqrt{\psi}\right)^2}{2(C-1)^2\sqrt{\psi}}\,. \tag{27}$$

Now we calculate the commutation relations between NC coordinates $\hat{x}_\mu$ and the Dirac derivatives D_ν:

$$\begin{aligned}
&[D_i,\hat{x}_j]=\delta_{ij}(-\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu})\,,\\
&[D_i,\hat{x}_n]=0\,,\\
&[D_n,\hat{x}_i]=\mathrm{i}aD_i\,,\\
&[D_n,\hat{x}_n]=\sqrt{1-a^2D_\mu D_\mu}\,.
\end{aligned} \tag{28}$$

These relations are universal for both realizations, $\psi=1$ and $\psi=1+2A$, and they involve only the deformation parameter a.

The corresponding coproduct is given by [16–18]

$$\begin{aligned}
\triangle D_n&=D_n\otimes\Big(-\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu}\Big)\\
&\quad+\frac{\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu}}{1-a^2D_kD_k}\otimes D_n\\
&\quad+\mathrm{i}aD_i\frac{\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu}}{1-a^2D_kD_k}\otimes D_i\,,\\
\triangle D_i&=D_i\otimes\Big(-\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu}\Big)+1\otimes D_i\,.
\end{aligned}$$

The Dirac derivatives, together with the generators of the rotations $M_{\mu\nu}$, form a κ-deformed Euclidean Hopf algebra which is undeformed in the algebra sector; see (7) and (21). The deformation is purely in the coalgebra sector following from (13), (14) and (28). Namely, the coalgebra structure is determined from the deformed commutation relations, (13), (14) and (28), in a simple and unique way.

In the first realization $\psi=1$ one finds the relation

$$\mathrm{e}^{-A}=-\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu}\,, \tag{29}$$

and in the second realization $\psi=1+2A$ the following relation holds:

$$\frac{1}{\sqrt{1+2A}}=-\mathrm{i}aD_n+\sqrt{1-a^2D_\mu D_\mu}\,. \tag{30}$$

Note that the relations $[D_\mu,\hat{x}_\nu]$, (28), and the vacuum condition $D_\mu|0\rangle=0$ define the Fock space. The Gramm matrices in the Fock space [20, 22] imply the relations $[\hat{x}_\mu,\hat{x}_\nu]$, (3), and $[D_\mu,D_\nu]=0$.

We point out that the κ-deformed NC space of the Lie algebra type with the structure constants $C_{\mu\nu\lambda}=a_\mu\delta_{\nu\lambda}-a_\nu\delta_{\mu\lambda}$ (see (1) and (2)),

$$[\hat{x}_\mu,\hat{x}_\nu]=\mathrm{i}C_{\mu\nu\lambda}\hat{x}_\lambda\,, \tag{31}$$

together with the undeformed $SO_a(n)$ rotation algebra, (7), leads to the relations

$$[M_{\mu\nu},\hat{x}_\lambda]=\hat{x}_\mu\delta_{\nu\lambda}-\hat{x}_\nu\delta_{\mu\lambda}-\mathrm{i}C_{\mu\nu\alpha}M_{\alpha\lambda}\,. \tag{32}$$

Demanding that the Dirac derivatives commute pairwise and transform under a vector representation of $SO_a(n)$, (21), we obtain a universal commutation relation:

$$[D_\mu,\hat{x}_\nu]=\delta_{\mu\nu}\sqrt{1-a^2D_\alpha D_\alpha}+\mathrm{i}C_{\mu\alpha\nu}D_\alpha\,. \tag{33}$$

Equations (12) and (28) are unified in the above equations (32) and (33). The Fock space representation is defined by $D_\mu|0\rangle=0$, $\forall\mu$.

The above NC space of the Lie algebra type with undeformed $SO_a(n)$ algebra, and commuting Dirac derivatives, transforming as a vector representation under $SO_a(n)$, and with a smooth limit to Euclidean space, is unique. The realizations of NC space defined by (7), (21), (31)–(33) and their properties will be treated separately. Some examples of the Poincaré invariant interpretation of NC spaces and twisted Poincaré coalgebra were also considered in [27–29].

Furthermore, it is interesting that, for $n=1$, the relation $[D_n,\hat{x}_n]$ becomes

$$[D,\hat{x}]=\sqrt{1-a^2D^2}\,.$$

In the quadratic approximation in a, $[D,\hat{x}]\approx 1-\frac{1}{2}a^2D^2$. This commutation relation leads to the generalized uncertainty relation with minimal length $\frac{|a|}{\sqrt{2}}$ [30].

6 Invariant operators

Analogously as we have defined Dirac derivatives, we introduce an invariant operator $\Box$, generalizing the Laplace (D'Alambert) operator, by the equation

$$[M_{\mu\nu},\Box]=0\,. \tag{34}$$

A simple Ansatz is

$$\Box=\Delta H_1(A)+\partial_n^2H_2(A)\,, \tag{35}$$

with boundary conditions

$$H_1(0) = 1, \quad H_2(0) = 1 .$$

Then, from the relation (34), we obtain two equations:

$$\begin{aligned} AF_2H_2' + 2F_2H_2 - 2F_1H_1 &= 0 , \\ F_2H_1' - 2(F_3+F_4)H_1 &= 0 . \end{aligned} \tag{36}$$

The solutions for the first realization $\psi = 1$ are

$$H_1 = \frac{e^{-A}}{\varphi^2}, \quad H_2 = -\frac{2[1-\cosh A]}{A^2} . \tag{37}$$

The second realization is for $\psi = 1+2A$

$$H_1 = \frac{1}{\sqrt{\psi}\varphi^2}, \quad H_2 = \frac{2}{A^2}\left(\frac{1+A}{\sqrt{1+2A}} - 1\right) . \tag{38}$$

Alternatively, we define another invariant operator

$$D_\mu D_\mu = D_iD_i + D_nD_n . \tag{39}$$

For both realizations, $\psi = 1$ and $\psi = 1+2A$, the universal relations

$$\begin{aligned} D_\mu D_\mu &= \Box(1 - \frac{a^2}{4}\Box) , \\ [\Box, \hat{x}_\mu] &= 2D_\mu \end{aligned} \tag{40}$$

hold, depending only on the deformation parameter a. The corresponding Leibniz rule for $\Box(fg)$ can easily be derived [18]. Note that there are infinitely many Dirac derivatives and Laplace operators, differing by multiplication by the function $\phi(a^2\Box)$ with $\phi(0) = 1$.

The new realizations can be summarized as follows.

I realization: $\psi = 1$

We have

$$\begin{aligned} M_{in} &= x_i\partial_n\varphi\frac{e^{2A}-1}{2A} - x_n\partial_i\frac{1}{\varphi} + \mathrm{i}ax_i\Delta\frac{1}{2\varphi} - \mathrm{i}ax_k\partial_k\partial_i\frac{\gamma}{\varphi} , \\ D_i &= \partial_i\frac{e^{-A}}{\varphi} , \qquad D_n = \partial_n\frac{\sinh A}{A} + \mathrm{i}a\Delta\frac{e^{-A}}{2\varphi^2} , \\ \Box &= \Delta\frac{e^{-A}}{\varphi^2} - \partial_n^2\frac{2[1-\cosh A]}{A^2} , \end{aligned} \tag{41}$$

where $\gamma = \frac{\varphi'}{\varphi} + 1$.

II realization: $\psi = 1+2A$

We have

$$\begin{aligned} M_{in} &= x_i\partial_n\varphi - x_n\partial_i\frac{\psi}{\varphi} + \mathrm{i}ax_i\Delta\frac{1}{2\varphi} - \mathrm{i}ax_k\partial_k\partial_i\frac{\gamma}{\varphi} , \\ D_i &= \partial_i\frac{C-\sqrt{\psi}}{(C-1)\sqrt{\psi}} , \\ D_n &= \partial_n\frac{1}{\sqrt{\psi}} + \mathrm{i}a\Delta\frac{(C-\sqrt{\psi})^2}{2(C-1)^2\sqrt{\psi}} , \\ \Box &= \Delta\frac{1}{\sqrt{\psi}\varphi^2} + \partial_n^2\frac{2}{A^2}\left(\frac{1+A}{\sqrt{1+2A}} - 1\right) , \end{aligned} \tag{42}$$

where

$$\varphi = \frac{C-1}{C-\sqrt{\psi}} , \quad \gamma = \psi\frac{\varphi'}{\varphi} + 1 , \quad C \in \mathbf{R}, \quad C \neq 1 .$$

We point out, that starting from the above two realizations we can construct infinitely many new realizations by similarity transformations (i.e., composing our realizations with inner automorphisms of the completed Weyl algebra of the x, ∂)

$$(\hat{x}_\mu)_\mathrm{S} = S\hat{x}_\mu S^{-1} , \qquad (M_{\mu\nu})_\mathrm{S} = SM_{\mu\nu}S^{-1} ,$$

where

$$S = \exp\{\Phi(a\partial_1, ..., a\partial_n)\} ,$$

with

$$\Phi(a\partial_1, ..., a\partial_n) = \sum_{\{m\}} c_{\{m\}}(x,\partial)\prod_{\mu=1}^{n}(a\partial_\mu)^{m_\mu}$$

and

$$[x_\mu\partial_\mu, c_{\{m\}}(x,\partial)] = 0 ,$$

where $\{m\} = (m_1, m_2, ..., m_n)$. To preserve the smooth limit $(\hat{x}_\mu)_\mathrm{S} \to x_\mu$ when $a \to 0$, the boundary condition on $\Phi(a\partial_1, ..., a\partial_n)$ has to be $\Phi(0, ..., 0) = 0$, i.e. $S \to 1$, when $a \to 0$. In this way one can obtain, for example, new solutions where φ, F, G, H depend not only on $A = \mathrm{i}a\partial_n$, but also on $\Delta = \partial_i\partial_i$.

Furthermore, two realizations with $\varphi_1(A)$ and $\varphi_2(A)$, but with $\psi_1(A) = \psi_2(A) = \psi(A)$, can be connected by S_{12}:

$$(\hat{x}_\mu)_2 = S_{12}(\hat{x}_\mu)_1S_{12}^{-1} ,$$

where

$$S_{12} = \exp\{x_i\partial_i(\ln\varphi_2 - \ln\varphi_1)\} .$$

7 Hermiticity

All relations of the type $[\hat{x}, \hat{x}]$, $[M, M]$, $[M, \hat{x}]$, $[M, D]$, $[D, D]$, $[D, \hat{x}]$, (3), (7), (12), (21) and (28), are invariant under the formal antilinear involution:

$$\hat{x}_\mu^\dagger = \hat{x}_\mu, \quad D_\mu^\dagger = -D_\mu, \quad M_{\mu\nu}^\dagger = -M_{\mu\nu}, \quad c^\dagger = \bar{c}, \quad c \in \mathbf{C} . \tag{43}$$

The order of elements in the product is inverted under the involution. The commutative coordinates x_μ and their derivatives ∂_μ also satisfy the involution property: $x_\mu^\dagger =$

x_μ, $\partial_\mu^\dagger = -\partial_\mu$. It is natural to ask whether the realizations, (17)–(20), satisfy the involution property, (43). It is easy to verify that $\hat{x}_i^\dagger = \hat{x}_i$, and $D_\mu^\dagger = -D_\mu$, $M_{ij}^\dagger = -M_{ij}$. However, generally, $\hat{x}_n^\dagger \neq \hat{x}_n$, $M_{in}^\dagger \neq -M_{in}$.

We point out that all realizations can be made hermitian, i.e. consistent with (43), by defining

$$\hat{x}_n^{\mathrm{h}} = \frac{1}{2}\left(\hat{x}_n + \hat{x}_n^\dagger\right) , \quad M_{in}^{\mathrm{a.h.}} = \frac{1}{2}\left(M_{in} - M_{in}^\dagger\right) . \tag{44}$$

All commutation relations are preserved by this redefinition of $\hat{x}_n$ and M_{in} in any realization. Namely, if $\hat{x}_\mu$, $M_{\mu\nu}$ is a realization of (3), (7) and (12), then $\hat{x}_\mu^\dagger$, $-M_{\mu\nu}^\dagger$ is also a realization of the same relations. Moreover, any linear combination $\alpha\hat{x}_\mu + (1-\alpha)\hat{x}_\mu$ and $\alpha M_{\mu\nu} - (1-\alpha)M_{\mu\nu}^\dagger$, $\alpha \in \mathbf{R}$ satisfies the same relations, (3), (7), and (12). Specially, for $\alpha = \frac{1}{2}$, we obtain a hermitian realization of the NC space defined by (3), (7) and (12). For example,

$$\hat{x}_n^{\mathrm{h}} = \frac{1}{2}\left(x_n\psi + \mathrm{i}ax_k\partial_k\gamma + \psi x_n + \mathrm{i}a\gamma\partial_k x_k\right) = x_n\psi + \mathrm{i}ax_k\partial_k\gamma + \frac{\mathrm{i}a}{2}\psi' + \frac{\mathrm{i}a}{2}(n-1)\gamma . \tag{45}$$

The simplest realization of κ-deformed space satisfying the hermiticity property (43) is the left realization with $\varphi = \mathrm{e}^{-A}$, $\psi = 1$ and $\gamma = 0$, i.e. (see Sect. 8)

$$\begin{aligned} &\hat{x}_i = x_i\mathrm{e}^{-A}, \qquad \hat{x}_n = x_n , \\ &D_i = \partial_i, \qquad D_n = \frac{1}{a}\sin(a\partial_n) + \frac{\mathrm{i}a}{2}\Delta\mathrm{e}^A , \\ &M_{ij} = x_i\partial_j - x_j\partial_i , \\ &M_{in} = \frac{1}{a}x_i\sin(a\partial_n) - x_n\partial_i\mathrm{e}^A + \frac{\mathrm{i}a}{2}x_i\Delta\mathrm{e}^A , \\ &\Box = -\frac{2}{a^2}[\cos(a\partial_n) - 1] + \Delta\mathrm{e}^A , \end{aligned} \tag{46}$$

with

$$\mathrm{e}^{\pm\mathrm{i}a\partial_n} f(x, ..., x_n) = f(x, ..., x_{n-1}, x_n \pm \mathrm{i}a) . \tag{47}$$

From the physical point of view, every φ-realization is allowed and, in some sense, corresponds to choosing a "gauge" for a concrete calculation. Moreover, non-hermitian realizations (not satisfying the hermiticity properties (43)) are allowed for concrete calculations [18]. For example, the symmetric realization $\varphi_{\mathrm{S}}(A) = \frac{A}{\mathrm{e}^A - 1}$ is not hermitian $\hat{x}_n^+ \neq \hat{x}_n$ [16–18] (see also Sect. 8, (67)).

8 Realizations of star products

There exists a vector space isomorphism between (the coordinate algebras of) the NC space $\mathbf{R}_a^n$ and the Euclidean space $\mathbf{R}^n$, depending on the function $\varphi(A)$ ($\psi = 1$ or $\psi = 1 + 2A$). In other words, for a given realization described by $\varphi(A)$ there is a unique mapping from the functions of the NC coordinates $\hat{x}_\mu$ to the functions of the commutative coordinates x_μ.

Let us define the "vacuum" state:

$$|0\rangle = 1, \qquad D_\mu|0\rangle = \partial_\mu|0\rangle = 0 . \tag{48}$$

Then we define a mapping from $f(\hat{x})$ to $f_\varphi(x)$ in a given φ-realization as

$$f(\hat{x}_\varphi)|0\rangle = f_\varphi(x) . \tag{49}$$

The functions $f(\hat{x})$ are defined as a formal power series in NC coordinates. Note that all monomials in which the $\hat{x}_1$ appear m_1 times, $\hat{x}_2$ m_2 times, ..., $\hat{x}_n$ m_n times, differ under permutations, i.e. there are $\begin{pmatrix} m \\ m_n \end{pmatrix}$ different monomials, where $m = \sum m_\mu$. However, they are proportional to each other. A basis in the space of monomials is fixed in a given φ-realization by

$$M_\varphi(\hat{x})|0\rangle = M_\varphi(x) + P_\varphi(x) , \tag{50}$$

where $M_\varphi(\hat{x})$ is a linear combination of monomials of the same type $(m_1, ..., m_n)$ (i.e. $\hat{x}_1$ appearing m_1 times, $\hat{x}_2$ – m_2 times, etc.), and $P_\varphi(x)$ is a polynomial of lower order than $M_\varphi(x)$. We generally write (50) as

$$\tilde{M}_\varphi(\hat{x})|0\rangle = \left[M_\varphi(\hat{x}) + \tilde{P}_\varphi(\hat{x})\right]|0\rangle = M_\varphi(x) ,$$

where $\tilde{P}_\varphi(\hat{x})$ is a polynomial in $\hat{x}$ of lower order than $M_\varphi(\hat{x})$. This means that a given φ-realization induces a natural basis for monomials, i.e. a natural ordering prescription; and vice versa, an ordering uniquely defines the φ-realization. For example, (4) and (50) imply $\prod \hat{x}_i^{m_i}|0\rangle = \prod x_i^{m_i}$ and for the mixed second-order monomials we obtain

$$\begin{aligned} M_\varphi(\hat{x}) &= [1 + \varphi'(0)]\hat{x}_i\hat{x}_n - \varphi'(0)\hat{x}_n\hat{x}_i\hat{x}_i\hat{x}_n \\ &\quad - \mathrm{i}a\varphi'(0)\hat{x}_i , \\ M_\varphi(\hat{x}_\varphi)|0\rangle &= x_i x_n . \end{aligned} \tag{51}$$

Let the $\tilde{M}_\varphi$ basis correspond to a given φ-realization. Then

$$f(\hat{x}) = f_\varphi(\hat{x}) = \sum c_\varphi\tilde{M}_\varphi(\hat{x}) , \quad f(\hat{x}_\varphi)|0\rangle = f_\varphi(\hat{x}_\varphi)|0\rangle = f_\varphi(x) . \tag{52}$$

Now we define a star product in a given φ-realization as

$$(f_\varphi \star_\varphi g_\varphi)(x) = f_\varphi(\hat{x}_\varphi)g_\varphi(\hat{x}_\varphi)|0\rangle f_\varphi(\hat{x}_\varphi)g_\varphi(x) . \tag{53}$$

Generally,

$$\begin{aligned} x_i \star_\varphi f(x) &= (\hat{x}_\varphi)_i f(x) = x_i\varphi(A)f(x) , \\ x_n \star_\varphi f(x)(\hat{x}_\varphi)_n f(x) &= [x_n\psi(A) + \mathrm{i}ax_k\partial_k\gamma(A)]f(x) \end{aligned} \tag{54}$$

and

$$\begin{aligned} f(x) \star_\varphi x_i &= x_i\varphi(A)\mathrm{e}^A f(x) , \\ f(x) \star_\varphi x_n &= [x_n\psi(A) + \mathrm{i}ax_k\partial_k(\gamma(A) - 1)]f(x) . \end{aligned} \tag{55}$$

Realizations with $\psi = 1$

We have for $\psi = 1$, $\gamma = \frac{\varphi'}{\varphi} + 1$, we find a closed form for the star product in the φ-realization:

$$(f \star g)(z) = \exp\left\{ z_i \partial_{x_i} \left[\frac{\varphi(A_x + A_y)}{\varphi(A_x)} - 1 \right] + z_i \partial_{y_i} \left[\frac{\varphi(A_x + A_y)}{\varphi(A_y)} \mathrm{e}^{A_x} - 1 \right] \right\} f(x) g(y) \Big|_{\substack{x=z \\ y=z}}, \tag{56}$$

where $A_x = \mathrm{i}a\frac{\partial}{\partial x_n}$ and $A_y = \mathrm{i}a\frac{\partial}{\partial y_n}$. From (56) it follows generally that

$$(g \star f)(z)\Big|_{\varphi(A)} = (f \star g)(z)\Big|_{\varphi(A)\mathrm{e}^A}. \tag{57}$$

Using (41), $\partial_i = D_i \varphi \mathrm{e}^A$ and the expression for coproduct $\triangle D_i$ (after (28)), we can write (56) for the star product as

$$(f \star g)(z) = \Big(m\{exp[z_i(\triangle - \triangle_0)\partial_i](f \otimes g)\}\Big)(x)\Big|_{x=z},$$

where $\triangle_0 \partial_i = \partial_i \otimes 1 + 1 \otimes \partial_i$ is the undeformed coproduct, and m is the multiplication map (product) in the Hopf algebra.

In the second order in a, from (56) we obtain

$$\begin{aligned}(f \star g)(z) = f(z)g(z) + \Big\{ & z_i \Big[\Big(1 + \varphi'(0)\Big) A_x \partial_{y_i} + \varphi'(0) \partial_{x_i} A_y \Big] \\ & + z_i \Bigg[\left(\frac{1}{2} + \varphi'(0) + \frac{1}{2}\varphi''(0) \right) A_x^2 \partial_{y_i} \\ & + \left(\varphi''(0) - (\varphi'(0))^2 \right) \left(\partial_{x_i} A_x A_y + A_x A_y \partial_{y_i} \right) \\ & + \frac{1}{2} \varphi''(0) \partial_{x_i} A_y^2 \Bigg] \\ & + \frac{1}{2} z_i z_j \Big[\Big(1 + \varphi'(0)\Big)^2 A_x^2 \partial_{y_i} \partial_{y_j} + 2\varphi'(0) \\ & \times \Big(1 + \varphi'(0)\Big) \partial_{x_i} A_x A_y \partial_{y_j} \\ & + \Big(\varphi'(0)\Big)^2 \partial_{x_i} \partial_{x_j} A_y^2 \Big] \Big\} f(x) g(y) \Big|_{\substack{x=z \\ y=z}} \\ & + \mathcal{O}(a^3)\end{aligned} \tag{58}$$

and consequently

$$\begin{aligned}\Big(f \star g - g \star f\Big)(x) = & \Big(A_x f(x)\Big)\Big(\mathcal{N}_x g(x)\Big) \\ & - \Big(\mathcal{N}_x f(x)\Big)\Big(A_x g(x)\Big) \\ & + \left(\frac{1}{2} + \varphi'(0)\right) \Big[\Big(A_x^2 f(x)\Big)\Big(\mathcal{N}_x^2 g(x)\Big) \\ & - \Big(\mathcal{N}_x^2 f(x)\Big)\Big(A_x^2 g(x)\Big) \Big] + \mathcal{O}(a^3),\end{aligned} \tag{59}$$

where $\mathcal{N}_x = x_i \frac{\partial}{\partial x_i}$. We point out that, generally, a factor $A \otimes \mathcal{N} - \mathcal{N} \otimes A$ appears in all orders of the expansion $(f \star g - g \star f)(x)$. For a given φ-realization there is a monomial basis M_φ satisfying (50) with $P_\varphi(x) = 0$, i.e. $M_\varphi(\hat{x}_\varphi)|0\rangle = M_\varphi(x)$, and vice versa. Let us consider three cases with $\psi = 1$.

Left ordering

If we define the M_φ basis in such a way that all $\hat{x}_n$ are at the most left in any monomial, then it follows from (4) and (50), and $\hat{x}_n^{m_n} \prod \hat{x}_i^{m_i} |0\rangle = x_n^{m_n} \prod x_i^{m_i}$, that $\hat{x}_n = x_n$, i.e. $\psi = 1$, $\gamma = 0$, $\varphi = \mathrm{e}^{-A}$. The star product for the left ordering is given by (56) with $\varphi_{\mathrm{L}} = \mathrm{e}^{-A}$, and alternatively by

$$(f \star_{\varphi_{\mathrm{L}}} g)(x) = \mathrm{e}^{-\mathrm{i}a x_i \partial_i^x \partial_n^y} f(x) g(y)\Big|_{y=x}. \tag{60}$$

Right ordering

Similarly, the right ordering is defined so that $\hat{x}_n$ are at the most right in any monomial. Then it follows from (4) and (50), and $(\prod \hat{x}_i^{m_i}) \hat{x}_n^{m_n} |0\rangle = (\prod x_i^{m_i}) x_n^{m_n}$, that $\hat{x}_i = x_i$, $\psi = 1$, $\varphi = 1$, $\gamma = 1$.

The star product for the right ordering is given by (56) with $\varphi_{\mathrm{R}} = 1$, and alternatively by

$$(f \star_{\varphi_{\mathrm{R}}} g)(x) = \mathrm{e}^{i a y_i \partial_i^y \partial_n^x} f(x) g(y)\Big|_{y=x}. \tag{61}$$

Symmetric ordering

The general series expansion formula for the star product of the Lie algebra type NC space, described by the structure constants $C_{\mu\nu\lambda}$, was given in the symmetric ordering [31]. The lowest-order symmetric monomials are

$$\begin{aligned}&\{\hat{x}_i \hat{x}_j, \frac{1}{2}(\hat{x}_i \hat{x}_n + \hat{x}_n \hat{x}_i), \hat{x}_n^2\}, \\ &\{\hat{x}_i \hat{x}_j \hat{x}_k, \frac{1}{3}(\hat{x}_i \hat{x}_j \hat{x}_n + \hat{x}_i \hat{x}_n \hat{x}_j + \hat{x}_n \hat{x}_i \hat{x}_j), \\ &\frac{1}{3}(\hat{x}_i \hat{x}_n^2 + \hat{x}_n \hat{x}_i \hat{x}_n + \hat{x}_n^2 \hat{x}_i), \hat{x}_n^3\}, \quad \text{etc}.\end{aligned}$$

To find a φ-realization corresponding to a symmetric ordering, we impose the condition (50) as follows:

$$\sum_\pi \pi M(\hat{x})|0\rangle = \binom{m}{m_n} M(x), \tag{62}$$

where the summation is over all different monomials $\pi M(\hat{x})$ differing by permutations (summed with equal weights). There are $\binom{m}{m_n}$ different monomials on the LHS, where $m = \sum m_\mu$. Specially, for $m = k+1$, $m_n = k$, we have

$$\sum_{r=0}^{k} (\hat{x}_\varphi)_n^r (\hat{x}_\varphi)_i (\hat{x}_\varphi)_n^{k-r} |0\rangle = (k+1) x_i x_n^k, \quad \forall k \in \mathbf{N} \tag{63}$$

and use the relation (obtained by shifting $\hat{x}_i$ to the left)

$$\sum_{r=0}^{k} \hat{x}_n^r \hat{x}_i \hat{x}_n^{k-r} = \sum_{r=0}^{k} \binom{k+1}{r+1} (\mathrm{i}a)^r \hat{x}_i \hat{x}_n^{k-r}, \quad (64)$$

from which we find

$$\sum_{r=0}^{k}\sum_{p=0}^{k-r} \binom{k+1}{r+1}\binom{k-r}{p} \times (\mathrm{i}a)^{r+p} \varphi^{(p)}(0) x_i x_n^{k-r-p} = (k+1) x_i x_n^k, \quad (65)$$

where $\varphi^{(p)}(0)$ is the pth derivative calculated at 0. For $l \geq 1$, we get

$$\sum_{p=0}^{l} \binom{l+1}{p} \varphi^{(p)}(0) = 0. \quad (66)$$

Solving the above recursive relations starting with $\varphi(0) = 1$, we obtain

$$\varphi'(0) = -\frac{1}{2}, \qquad \varphi''(0) = \frac{1}{12}, \qquad \varphi'''(0) = 0, \quad \text{etc.}$$
$$\varphi_{\mathrm{S}}(A) = \sum_{p=0}^{\infty} \frac{\varphi_{\mathrm{S}}^{(p)}(0)}{p!} A^p = \frac{A}{\mathrm{e}^A - 1}. \quad (67)$$

One can show that (62) is satisfied for every symmetrically ordered monomial with the above φ_{S}, (67). Note that

$$\gamma_{\mathrm{S}}(A) = \frac{\varphi'_{\mathrm{S}}(A)}{\varphi_{\mathrm{S}}(A)} + 1 = -\frac{\varphi_{\mathrm{S}}(A) - 1}{A},$$
$$\varphi_{\mathrm{S}}(A)\mathrm{e}^A = \varphi_{\mathrm{S}}(-A), \qquad \gamma_{\mathrm{S}}(A) + \gamma_{\mathrm{S}}(-A) = 1,$$
$$g(x) \star_{\varphi_{\mathrm{S}}} f(x)\Big|_a = f(x) \star_{\varphi_{\mathrm{S}}} g(x)\Big|_{-a}. \quad (68)$$

Inserting $\varphi_{\mathrm{L}}(A) = \mathrm{e}^{-A}$, $\varphi_{\mathrm{R}}(A) = 1$, $\varphi_{\mathrm{S}}(A) = \frac{A}{\mathrm{e}^A - 1}$ and the corresponding $\psi = 1$, $\gamma = \frac{\varphi'}{\varphi} + 1$ functions for the left, right and symmetric ordering, we obtain the results for these three special cases [18].

Generally, if $\psi = 1$ then $\tilde{M}_\varphi(\hat{x}) = M_\varphi(\hat{x})$, i.e. $P_\varphi(x) = 0$, (4) and (50).

Realizations with $\psi = 1 + 2A$

For the realizations with $\psi = 1 + 2A$, the condition described by (50),

$$\tilde{M}_\varphi(\hat{x})|0\rangle = M_\varphi(x),$$

can be fulfilled generally if $\tilde{M}_\varphi(\hat{x}) \neq M_\varphi(\hat{x})$, i.e. if $P_\varphi(x) \neq 0$, (50). Namely,

$$\hat{x}_n^k|0\rangle = x_n[(1+2A)x_n]^{k-1} = x_n^k + P_{k-1}(x), \qquad k \geq 1, \quad (69)$$

where $k \geq 1$, and $P_{k-1}(x)$ is a polynomial of order $(k-1)$.

Generally, this holds for all realizations with $\psi \neq 1$, including the hermitian realizations with $\psi = 1$, satisfying (43). For example, we obtain (see (45))

$$\hat{x}_n^{\mathrm{h}}|0\rangle = x_n + \frac{\mathrm{i}a}{2}\psi'(0) + \frac{\mathrm{i}a}{2}(n-1)\gamma(0) \neq x_n. \quad (70)$$

The isomorphism f to f_φ is defined by $f(\hat{x}_\varphi)|0\rangle = f_\varphi(x)$, (49) and (52), and the corresponding star product is defined by (53).

Our approach can be applied and extended to κ-deformed Minkowski space. One can define the Klein–Gordon and Dirac equations for free fields and gauge theory in κ-deformed space for an arbitrary φ-realization. There are still some open problems concerning the invariant integral and the variational principle [32] that will be treated separately.

9 Conclusion

We have presented a unified and simple method of constructing realizations of NC spaces in terms of commutative coordinates x_μ and their derivatives ∂_μ of Euclidean space. This method can also be applied to spaces with arbitrary signatures, especially to Minkowski-type spaces.

Particularly, we have studied κ-deformed Euclidean space with undeformed rotation algebra $SO_a(n)$. Dirac derivatives are constructed as a vector representation under $SO_a(n)$, and they commute themselves, $[D_\mu, D_\nu] = 0$. Similarly, there is an invariant operator $\Box$ such that $[M_{\mu\nu}, \Box] = 0$. We have found two infinite new families of realizations described by $\psi = 1$ and $\psi = 1 + 2A$ ($\psi = 1$, φ arbitrary and $\psi = 1 + 2A$, $\varphi = \frac{C-1}{C-\sqrt{\psi}}$, $C \neq 1$).

Furthermore, we have shown how these realizations can be extended to satisfy the hermitian properties (43).

We have constructed the star product for any realization. We point out that all realizations are on an equal footing and any of the realizations can be used for a concrete physical calculation, and its meaning is similar to the case when a particular gauge is chosen.

Finally, we have constructed κ-deformed NC space of the Lie algebra type, (31) with undeformed Poincaré algebra (7) and (21) and deformed coalgebra (32) and (33), in a unique way. Our approach may be useful in quantum-gravity models, specially in 2+1 dimension. In this case, the corresponding Lie algebra is $SU(2)$ or $SU(1,1)$ [33–35]. For general Lie algebra, (1), there exists a universal formula for the mapping $\hat{x}_\mu = x_\alpha \varphi_{\alpha\mu}(\partial_1, ..., \partial_n)$, corresponding to the totally symmetric ordering [36].

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